

2019 考研数学三答案解析

15、【答案】 $f'(x) = \begin{cases} 2x^{2x}(\ln x + 1); & x > 0 \\ e^x(x + 1); & x < 0 \end{cases}$

极大值 $f(0) = 1$, 极小值 $f(-1) = 1 - \frac{1}{e}$, $f(\frac{1}{e}) = e^{-\frac{2}{e}}$.

【解析】解：当 $x > 0$ 时：

$$f'(x) = (e^{2x \ln x} - 1)' = (e^{2x \ln x})' = e^{2x \ln x} (2 \ln x + 2) = 2x^{2x} (\ln x + 1)$$

当 $x < 0$ ：

$$f'(x) = e^x + x e^x = e^x (x + 1)$$

因此 $f'(x) = \begin{cases} 2x^{2x}(\ln x + 1); & x > 0 \\ e^x(x + 1); & x < 0 \end{cases}$

当 $x = 0$ ：

$$f'_+(0) = \lim_{x \rightarrow 0^+} \frac{f(x) - f(0)}{x} = \lim_{x \rightarrow 0^+} \frac{x^{2x} - 1}{x} = \lim_{x \rightarrow 0^+} \frac{e^{2x \ln x} - 1}{x} = \lim_{x \rightarrow 0^+} \frac{2x \ln x}{x} = -\infty$$

$$f'_-(0) = \lim_{x \rightarrow 0^-} \frac{f(x) - f(0)}{x} = \lim_{x \rightarrow 0^-} \frac{x e^x}{x} = \lim_{x \rightarrow 0^-} e^x = 0$$

当 $x > 0$ 时, $f'(0) < 0$, $f(x)$ 单调递减, 当 $x < 0$ 时, $f'(0) > 0$, $f(x)$ 单调递增

因此 $f(x)$ 在 $x = 0$ 处取得极大值, 且 $f(0) = 1$.

令 $f'(x) = 0$ 得, $x = -1$ 及 $x = \frac{1}{e}$. 又 $f''(-1) > 0$, $f''(\frac{1}{e}) > 0$, 故极小值为 $f(-1) = 1 - \frac{1}{e}$, $f(\frac{1}{e}) = e^{-\frac{2}{e}}$.

16、【答案】 $1 - 3f''_{11} - f''_{22}$.

【解析】依题意知,

$$\frac{\partial g}{\partial x} = y - f'_1(x+y, x-y) - f'_2(x+y, x-y),$$

$$\frac{\partial g}{\partial y} = x - f'_1(x+y, x-y) + f'_2(x+y, x-y).$$

因为 $f(u, v)$ 具有二阶连续偏导数, 故 $f_{12}'' = f_{21}''$, 因此,

$$\frac{\partial^2 g}{\partial x^2} = -(f_{11}'' + f_{12}'') - (f_{21}'' + f_{22}'') = -f_{11}'' - 2f_{12}'' - f_{22}'' ,$$

$$\frac{\partial^2 g}{\partial x \partial y} = 1 - (f_{11}'' - f_{12}'') - (f_{21}'' - f_{22}'') = 1 - f_{11}'' + f_{22}'' ,$$

$$\frac{\partial^2 g}{\partial y^2} = -(f_{11}'' - f_{12}'') + (f_{21}'' - f_{22}'') = -f_{11}'' + 2f_{12}'' - f_{22}'' .$$

所以, $\frac{\partial^2 g}{\partial x^2} + \frac{\partial^2 g}{\partial x \partial y} + \frac{\partial^2 g}{\partial y^2} = 1 - 3f_{11}'' - f_{22}''$.

17、

【答案】(1) $y(x) = \sqrt{x}e^{\frac{x^2}{2}}$. (2)

【解析】(1) $y(x) = e^{-\int -x dx} \left(C + \int \frac{1}{2\sqrt{x}} e^{\frac{x^2}{2}} \cdot e^{\int -x dx} \right) = e^{\frac{x^2}{2}} (C + \sqrt{x})$.

因为 $y(1) = \sqrt{e}$, 故 $C = 0$, 所以 $y(x) = \sqrt{x}e^{\frac{x^2}{2}}$.

(2) 由旋转体体积公式,

$$V = \pi \int_1^2 (\sqrt{x}e^{\frac{x^2}{2}})^2 dx = \pi \int_1^2 xe^{x^2} dx = \frac{\pi}{2}(e^4 - e).$$

18、【答案】设在区间 $[n\pi, (n+1)\pi]$ ($n = 0, 1, 2, \dots$) 上所围的面积记为 u_n , 则

$$u_n = \int_{n\pi}^{(n+1)\pi} e^{-x} |\sin x| dx = (-1)^n \int_{n\pi}^{(n+1)\pi} e^{-x} \sin x dx;$$

记 $I = \int e^{-x} \sin x dx$, 则 $I = -\int e^{-x} d \cos x = -(e^{-x} \cos x - \int \cos x de^{-x})$

$$= -e^{-x} \cos x - \int e^{-x} d \sin x = -e^{-x} \cos x - (e^{-x} \sin x - \int \sin x de^{-x})$$

$$= -e^{-x}(\cos x + \sin x) - I ,$$

所以 $I = -\frac{1}{2}e^{-x}(\cos x + \sin x) + C$;

因此 $u_n = (-1)^n \left(-\frac{1}{2}\right) e^{-x}(\cos x + \sin x) \Big|_{n\pi}^{(n+1)\pi} = \frac{1}{2}(e^{-(n+1)\pi} + e^{-n\pi})$;

(这里需要注意 $\cos n\pi = (-1)^n$)

$$\text{因此所求面积为 } \sum_{n=0}^{\infty} u_n = \frac{1}{2} + \sum_{n=1}^{\infty} e^{-n\pi} = \frac{1}{2} + \frac{e^{-\pi}}{1-e^{-\pi}} = \frac{1}{2} + \frac{1}{e^{\pi}-1}.$$

19、

【答案】(1)证明: $a_{n+1} - a_n = \int_0^1 x^n(x-1)\sqrt{1-x^2} dx < 0$, 所以 $\{a_n\}$ 单调递减.

$$a_n = -\frac{1}{3} \int_0^1 x^{n-1} d(1-x^2)^{\frac{3}{2}} = -\frac{1}{3} [x^{n-1}(1-x^2)^{\frac{3}{2}}]_0^1 - \int_0^1 (1-x^2)^{\frac{3}{2}} dx^{n-1}$$

$$= \frac{n-1}{3} \int_0^1 x^{n-2}(1-x^2)\sqrt{1-x^2} dx$$

$$= \frac{n-1}{3} (\int_0^1 x^{n-2}\sqrt{1-x^2} dx - \int_0^1 x^n\sqrt{1-x^2} dx)$$

$$= \frac{n-1}{3} (a_{n-2} - a_n),$$

$$\text{从而有 } a_n = \frac{n-1}{n+2} a_{n-2} \quad (n=2,3,\dots);$$

(2) 因为 $\frac{a_n}{a_{n-2}} < \frac{a_n}{a_{n-1}} < \frac{a_n}{a_n} = 1$, 而 $\lim_{n \rightarrow \infty} \frac{a_n}{a_{n-2}} = \lim_{n \rightarrow \infty} \frac{n-1}{n+2} = 1$, 由夹逼准则知

$$\lim_{n \rightarrow \infty} \frac{a_n}{a_{n-1}} = 1.$$

20、

【答案】 $a \neq -1$;

$$a=1 \text{ 时, } \beta_3 = (3-k)\alpha_1 + (-2+k)\alpha_2 + k\alpha_3 \quad (k \text{ 为任意常数});$$

$$\text{当 } a \neq \pm 1 \text{ 时, } \beta_3 = \alpha_1 - \alpha_2 + \alpha_3.$$

【解析】 令 $A = (\alpha_1, \alpha_2, \alpha_3)$, $B = (\beta_1, \beta_2, \beta_3)$, 所以, $|A| = 1 - a^2$, $|B| = 2(a^2 - 1)$.

因向量组 I 与 II 等价, 故 $r(A) = r(B) = r(A, B)$, 对矩阵 (A, B) 作初等行变换. 因为

$$(\mathbf{A}, \mathbf{B}) = \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 2 & 1 & 2 & 3 \\ 4 & 4 & a^2+3 & a+3 & 1-a & a^2+3 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 & 2 & 2 \\ 0 & 0 & a^2-1 & a-1 & 1-a & a^2-1 \end{pmatrix}.$$

当 $a=1$ 时, $r(\mathbf{A})=r(\mathbf{B})=r(\mathbf{A}, \mathbf{B})=2$; 当 $a=-1$ 时, $r(\mathbf{A})=r(\mathbf{B})=2$, 但 $r(\mathbf{A}, \mathbf{B})=3$; 当 $a \neq \pm 1$ 时, $r(\mathbf{A})=r(\mathbf{B})=r(\mathbf{A}, \mathbf{B})=3$. 综上, 只需 $a \neq -1$ 即可.

因为对列向量组构成的矩阵作初等行变换, 不改变线性关系.

$$\textcircled{1} \text{ 当 } a=1 \text{ 时, } (\alpha_1, \alpha_2, \alpha_3, \beta_3) \rightarrow \begin{pmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \text{ 故 } \beta_3 = x_1 \alpha_1 + x_2 \alpha_2 + x_3 \alpha_3 \text{ 的等价方程组为 } \begin{cases} x_1 = 3 - 2x_3, \\ x_2 = -2 + x_3. \end{cases}$$

故 $\beta_3 = (3-k)\alpha_1 + (-2+k)\alpha_2 + k\alpha_3$ (k 为任意常数);

$$\textcircled{2} \text{ 当 } a \neq \pm 1 \text{ 时, } (\alpha_1, \alpha_2, \alpha_3, \beta_3) \rightarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{pmatrix}, \text{ 所以 } \beta_3 = \alpha_1 - \alpha_2 + \alpha_3.$$

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解: (1) 相似矩阵有相同的特征值, 因此有 $\begin{cases} -2+x-2=2-1+y, \\ |A|=|B|, \end{cases}$

又 $|A| = -2(4-2x)$, $|B| = -2y$, 所以 $x=3, y=-2$.

(2) 易知 B 的特征值为 $2, -1, -2$; 因此

$$\mathbf{A} - 2\mathbf{E} \xrightarrow{r} \begin{pmatrix} 2 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \text{ 取 } \xi_1 = (-1, 2, 0)^T,$$

$$\mathbf{A} + \mathbf{E} \xrightarrow{r} \begin{pmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \text{ 取 } \xi_2 = (-2, 1, 0)^T,$$

$$\mathbf{A} + 2\mathbf{E} \xrightarrow{r} \begin{pmatrix} 4 & 0 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix}, \text{ 取 } \xi_3 = (-1, 2, 4)^T$$

令 $P_1 = (\xi_1, \xi_2, \xi_3)$, 则有 $P_1^{-1}AP_1 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$;

同理可得, 对于矩阵 B , 有矩阵 $P_2 = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $P_2^{-1}BP_2 = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$, 所以

$P_1^{-1}AP_1 = P_2^{-1}BP_2$, 即 $B = P_2P_1^{-1}AP_1P_2^{-1}$, 所以

$$P = P_1P_2^{-1} = \begin{pmatrix} -1 & -1 & -1 \\ 2 & 1 & 2 \\ 0 & 0 & 4 \end{pmatrix}.$$

22、

【答案】(1) $f_Z(z) = \begin{cases} pe^z, & z < 0, \\ (1-p)e^{-z}, & z \geq 0. \end{cases}$ (2) $p = \frac{1}{2}$; (3) 不独立.

【解析】(1) Z 的分布函数为 $F_Z(z) = P(XY \leq z) = P(Y = -1, X \leq -z) + P(Y = 1, X \leq z)$, 因为 X 与 Y 相互独立, 且 X 的分布函数为

$$F_X(x) = \begin{cases} 1 - e^{-x}, & x > 0, \\ 0, & x \leq 0. \end{cases}$$

因此, $F_Z(z) = p[1 - F_X(-z)] + (1-p)F_X(z) = \begin{cases} pe^z, & z < 0, \\ (1-p)(1 - e^{-z}), & z \geq 0. \end{cases}$

所以, Z 的概率密度为 $f_Z(z) = F'_Z(z) = \begin{cases} pe^z, & z < 0, \\ (1-p)e^{-z}, & z \geq 0. \end{cases}$

(2) 当 $Cov(X, Z) = EXZ - EX \cdot EZ = EX^2 \cdot EY - (EX)^2 \cdot EY = DX \cdot EY = 0$ 时, X 与 Z 不相关. 因为 $DX = 1$, $EY = 1 - 2p$, 故 $p = \frac{1}{2}$.

(3) 不独立. 因为

$$P(0 \leq X < 1, Z \leq 1) = P(0 \leq X < 1, XY \leq 1) = P(0 \leq X < 1),$$

而 $P(Z \leq 1) = F_Z(1) = (1-p)(1 - e^{-1}) \neq 1$, 故 $P(0 \leq X < 1, Z \leq 1) \neq P(0 \leq X < 1) \cdot P(Z \leq 1)$,

所以 X 与 Z 不独立.

23.

【解答】(1) 由密度函数的规范性可知 $\int_{-\infty}^{+\infty} f(x)dx = 1$, 即

$$\int_{\mu}^{+\infty} \frac{A}{\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \frac{A}{\sigma} \int_0^{+\infty} e^{-\frac{t^2}{2\sigma^2}} dt = \frac{\sqrt{2\pi}A}{2} \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{t^2}{2\sigma^2}} dt = A\sqrt{\frac{\pi}{2}} = 1 ,$$

$$\text{得 } A = \sqrt{\frac{2}{\pi}} .$$

$$(2) \text{ 设似然函数 } L(\sigma^2) = \prod_{i=1}^n f(x_i; \sigma^2) = \prod_{i=1}^n \sqrt{\frac{2}{\pi}} \frac{1}{\sigma} e^{-\frac{(x_i-\mu)^2}{2\sigma^2}} ,$$

$$\text{取对数 } \ln L(\sigma^2) = \sum_{i=1}^n \left[\ln \sqrt{\frac{2}{\pi}} - \frac{1}{2} \ln \sigma^2 - \frac{(x_i-\mu)^2}{2\sigma^2} \right] ;$$

$$\text{求导数 } \frac{d \ln L(\sigma^2)}{d\sigma^2} = \sum_{i=1}^n \left[-\frac{1}{2\sigma^2} + \frac{(x_i-\mu)^2}{2\sigma^4} \right] = -\frac{n}{2\sigma^2} + \frac{\sum_{i=1}^n (x_i-\mu)^2}{2\sigma^4} ,$$

$$\text{令导数为零解得 } \sigma^2 = \frac{1}{n} \sum_{i=1}^n (x_i-\mu)^2 ,$$

$$\text{故 } \sigma^2 \text{ 的最大似然估计量为 } \hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^n (X_i - \mu)^2 .$$